DC STABILITY OF NONLINEAR SYSTEMS

Z. BIOLEK*), D. BIOLEK)**
*) SPSE ROZNOV P.R., CZECH REPUBLIC
**) UTKO, BUT BRNO, CZECH REPUBLIC

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ABSTRACT: The paper points out some drawbacks of classical definitions of DC circuit and DC stability. New thermodynamic definitions are proposed. Stability testing of DC operating points is then based on the energetic description and can be easily implemented as computer algorithm.

INTRODUCTION

In the design of new circuit structures or integrated circuits, some problems can be caused by so-called unstable equilibrium states. All the quiescent states are given by the solution of algebraic equations of circuit with all inductors shunted and all capacitors disconnected. DC stable equilibrium states are called stable operating points. The term DC stability denotes circuit ability to eliminate slow fluctuations around the operating point, which can be caused by voltage or temperature dithering, by gradual modification of parameters of circuit elements, etc. If the circuit trajectory approaches such a stable operating point, the circuit is „snapped” to it.

On the contrary, unstable operating points amplify every slow deflection and push the trajectory off. Such unstable points can originate on purpose (e.g. in the flip-flop circuits) or inadvertently by incorrect design (topological error, neglecting a substantial nonlinearity, etc.). To find all the operating points is not as complicated as to identify which of them are unstable. The time, which plays an important role in the classical theory of stability, is not present in the mathematical models of circuits with excluded reactances.

A possible solution consists in extending a DC resistive circuit by auxiliary (parasitic) reactance elements and moving the stability problem to the known area of stability testing of dynamic circuits.

It is necessary to specify the term “circuit extension” by reactances. Apparently, it would be necessary – among others - to disconnect each branch and include here an inductance, and to mutually interconnect all nodes by capacitances. Such extension is problematic from the point of view of practical testing. In addition, it is not obvious how to realize it for macros described by various linking conditions. Operational amplifiers would have to be completed with an “appropriate” number of poles, two-port parameters would have to be frequency-dependent, etc.

In the papers by Green and Wilson [1], [2], where auxiliary reactances are considered, the way of their incorporation is not specified. In [3] parasitic capacitances are included between each node and the ground, and parasitic inductances in series with each resistor. Only “parasitic shunt capacitors and series inductors" are discussed in [1], [4], and [5].

Even more equivocations appear in (in)stability definitions. Some authors do not distinguish between the terms “operating point” and “equilibrium point”. A consistent distinction is given in [1]: the equilibrium point concerns a general dynamic circuit, the operating point describes the state of adjoint DC circuit. In [3], instead of “operating point” the terms "dc circuit's equilibrium point" or "resistive circuit's equilibrium point" are used.

The following terms appear in the literature in connection with the stability of both the operating and equilibrium points:

- stable equilibrium point and unstable equilibrium point [3] (commonly accepted definition),
- potentially stable operating point and unstable operating point [1],
- structurally stable equilibrium point and unstable equilibrium point [4],
- totally stable operating point [3],
- *k*-th order saddle-node unstable operating point [3]
- occasionally unstable operating point [3].
- stable, unstable, and conditionally stable operating point [6].
- asymptotically stable equilibrium point [7].

In all the cases, the stability test is performed above the linearized model of nonlinear circuit around the investigated operating point.

It is evident that there are more approaches to understanding DC stability. Below we summarize their common features in the classical definition of DC circuit as a limit case of dynamic circuit. We refer to the principal problems related to this conception of DC stability and to the method of testing it. We offer outlets based on the thermodynamic definition of DC circuit, the ensuing new conception of DC stability, and new methods of testing it.

With respect to the limited scope of contribution and for the sake of better readability, some propositions are given without proof. Details can be found in the literature [8-16].

DEFINITIONS AND BASIC TERMS

Classical definitions

- ✓ *DC circuit is a limiting case of real circuit with reactances, where the vector of reactance elements tends to zero.*

An advantage of this approach is that it means precisely what one intuitively understands as a DC circuit. In DC equilibrium state, true reactance elements do not find application. All the quiescent states are solutions of algebraic equations of circuit, in which all inductors are shunted and all capacitors are disconnected. The "classical" definition of DC circuit leads to the "classical" conception of DC stability:

- ✓ *Circuit is DC stable, if it is stable for an arbitrary infinitesimal vector of reactance parameters.*
- ✓ *Circuit is DC potentially stable, if it is stable for some of infinitesimal vectors of reactance parameters, and unstable for some others.*
- ✓ *Circuit is DC unstable, if it is unstable for an arbitrary infinitesimal vector of reactance parameters.*

It is evident that the above definitions are important for understanding the principle of DC stability and its relation to the general stability of dynamic circuit. Among others, one conclusion is that DC stability is a necessary but not sufficient condition of stability of the associated dynamic circuit. Conversely, a DC unstable circuit cannot be stabilized by augmenting it by other reactance elements (i.e. it cannot be frequency compensated). It means simultaneously that this kind of instability is caused not by the dynamic but by the DC feedback. Information about DC instability is thus contained in the circuit topology and in the circuit parameters. To obtain this information, we basically need neither the time factor nor the classical methods of stability testing. Following in these considerations, we find out that the classical definition of DC stability is not very suitable for rigorous testing. What do "classical" definitions offer in the area of practical stability testing?

The most reliable method of stability testing would be probably the following one: we analyze a circuit model which includes – on the level of lumped parameters – all reactance elements, i.e. parasitic capacitances and inductances of all circuit components and connecting wires, all internode capacitances, inductances in each branch, a large amount of mutual inductances, etc. The principal problem is that it is practically impossible to work with such a complicated model. That is why we almost always consider most parasitic reactances to be zero. In other words, we start the stability test by passing the vector of reactance elements to zero, but in an improper way. Neglecting parasitic capacitance or inductance can lead at the very beginning to the exclusion of just the only substantial parameter which causes instability. Parameter neglect can start troubles which are well known in the area of modelling. The circuit order degrades successively and its properties change discontinuously. Moreover, the direction in which the parameter vector moves to zero can be in general important. That is why a DC circuit as a limit - according to the classical definition- need not exist. Neglecting a parameter that is essential for stable behaviour can cause hardly explicable paradoxes [8-10].

In virtue of the above reasons, the „classical" definitions of DC circuit and DC stability do not appear to be a good starting point of practical stability testing. We would have to know a general approach to extending the circuit by "cardinal" reactances, and then to perform the classical test of stability.

As mentioned above, information about DC instability is hidden in circuit topology and in its DC parameters. If we identify only unstable operating points, we dispense with circuit extension by parasitic reactances because they do not affect DC instability. Below we propose a set of thermodynamic definitions, which take this reality into account.

Thermodynamic definitions

- ✓ *DC circuit is a thermodynamic system described by generalized fluxes I (electric currents) and generalized forces V (electric voltages).*

In this conception of DC circuit, we completely dispense with both reactance elements and classical dynamics. Instead, we use so-called virtual dynamics, which is based on the variation of circuit variables and on the energetic approach. This approach starts from the principle of minimum entropy production [11], which is valid for thermodynamic systems. Using this principle, the definition of DC stability can be as follows [10]:

- ✓ *Operating point is DC stable, if an arbitrary virtual deflection from this point increases circuit energy , i.e.*

$$\delta P(\delta \vec{y}) > 0, \qquad (1)$$

where $\delta \vec{y}$ is a virtual state deflection from equilibrium and δP is the power delivered to the circuit by the source of this deflection.

The thermodynamic conception of DC circuit and DC stability also offers a simple method of stability testing, which can be easily implemented as a computer algorithm. The circuit is deflected from its equilibrium in a general direction. The power delivered to the circuit from the source of deflection is calculated. To achieve DC stability, this power must be always positive according to equation (1). Practical testing consists in localizing the eigenvalues of so-called virtual matrix in the complex plane [12-16].

DC STABILITY REGIONS AND DC INSTABILITY

Let $\vec{y}$ be a vector describing the circuit state, and $\vec{y}_0$ the operating point under testing. The thermodynamic test of DC stability is based on observing energetic relations while the state is virtually changed from starting point $\vec{y}_0$ to the final one $\vec{y} = \vec{y}_0 + \delta\,\vec{y}$. In case of DC stability, an arbitrary virtual change of state with respect to the equilibrium state has to increase system energy. The DC stable operating point will lie in the local minimum of the energetic function [11].

The circuit will be deflected from its operating point. Power $\delta\,P$ delivered to the circuit will be registered for every deflection. The equilibrium state is DC stable only when inequality (1) is true for all infinitesimal deflections $\delta\,\vec{y}$.

It remains to solve the problem of how to realize deflections $\delta\,\vec{y}$ from the operating point. We place sources of fluctuations $\delta\,\vec{x}$ in the circuit in such a way as to examine the entire environment of the operating point. We speak about the *complete set of deflection sources*. Then inequality (1) $\delta\,P(\delta\,\vec{x}) > 0$ is the necessary and sufficient condition of DC stability of operating point $\vec{y}_0$. If sources $\delta\,\vec{x}$ had been chosen and placed such that they would not examine the entire environment of operating point $\vec{y}_0$, then condition (1) would be only the necessary but not sufficient condition of DC stability.

For a truly reliable examination of system behaviour around the equilibrium state, we would probably include all the conceivable fluctuation sources in the tested circuit. Consider a finite number of nodes and branches in the circuit. We include a source of voltage fluctuation in each branch and a source of current fluctuation between each two nodes, as illustrated in the upper part of Fig. 1. Considering ***n*** independent nodes, the total number P_v of voltage (P_i of current) sources of fluctuations will be as follows:

$$P_v = P_i = n(n+1)/2\,.$$

Obviously, the excitation of the same circuit by ***n(n-1)*** different sources of voltage and current deflections according to the upper part of Fig. 1, and by ***n*** current sources according to the bottom part of Fig. 1, will be equivalent on the assumption that

$$\delta\,I_M = \sum_N \left(\delta\,I_{MN} + \delta\,V_{MN} \cdot g_{MN}\right), \tag{2a}$$

$$\delta\,I_N = \sum_M \left(\delta\,I_{MN} + \delta\,V_{MN} \cdot g_{MN}\right), \tag{2b}$$

where g_{MN} is differential conductance of the branch *M-N*. Summation is accomplished according to all the other nodes, and the following rules formally hold:

$$\delta\,I_{MN} = -\delta\,I_{NM} \text{ and } \delta\,V_{MN} = -\delta\,V_{NM}\,.$$

The following important conclusion results from Fig. 1 and equations (2*a*) and (2*b*): the circuit state represented by the vector of nodal voltages $\vec{V}$ can be reliably swept in the entire environment of operating point $\vec{V}_0$ by the vector of current deflections $\vec{I}$ flowing into the independent nodes. This is also a complete set of deflection sources. It holds at the same time that a number of current fluctuations smaller than ***n*** does not examine the entire environment of operating point $\vec{V}_0$.

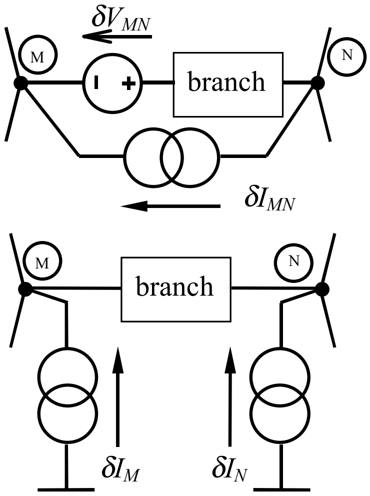

Fig. 1. *Equivalent methods of inserting deflection sources in the circuit.*

If the circuit is linearized around the operating point, then the stability of zero state is examined. Nodal voltages will be simultaneously the deflections from the operating point, otherwise $\vec{V} = \delta\,\vec{V}$, and they will depend on the deflection sources according to the law

$$\underline{G} \cdot \delta\,\vec{V} = \delta\,\vec{I}\,, \tag{3}$$

where $\underline{G}$ is the circuit conductivity matrix. In this case, matrix $\underline{G}$ is simultaneously the virtual matrix, which determines the circuit voltage reaction to the virtual current deflection. This matrix includes information about circuit topology and parameters, and thus information about DC stability.

As results from equation (3), the power delivered to the circuit by an arbitrary source of deflection from the operating point is

$$\delta\,P = \delta\,\vec{V}^T \cdot \delta\,\vec{I} = \delta\,\vec{V}^T \cdot \underline{G} \cdot \delta\,\vec{V}\,.$$

This quadratic form has to be positive definite in case of DC stability. This is accomplished if a general asymmetrical virtual matrix has all eigenvalues λ_i in the right-side complex plane.

DC STABILITY OF NONLINEAR SYSTEMS

General remarks

The hitherto considerations were based on the circuit model obtained after linearization around the investigated operating point. A circuit containing semiconductor devices can have more operating points. Some of them are unstable and we need not have any idea of their existence. In 1994, Green showed in [2] that at least one half of all the operating points are stable. The other operating points are unstable. They have the following peculiarities:

- *They are "unobservable", or hidden, their existence need not be apparent during regular circuit function.*
- *Although they repel circuit trajectories, numerical algorithms for their finding usually well converge to them.*
- *Instability cannot be frequency-compensated.*
- *In case of inadequate circuit extension by reactance elements, the classical "pole-location" stability criteria can fail.*
- *Even if they are "unobservable", they affect the overall circuit behaviour (transient to the stable operating point; they determine the necessity to use starting circuits; they accelerate trajectory to an undesirable operating point; etc.).*
- *Some of subsequent analyses lose their direct physical meaning (AC analysis).*

DC analogy of the First Lyapunov method

The first Lyapunov method shows when and why one can investigate the stability of nonlinear circuit by means of its linearized model. Stability is defined here as a circuit ability to come back to the equilibrium state after termination of a disturbance, which was the cause of the state deflection. The forms of trajectories, along which the circuit comes back, are given in the close neighbourhood of stable equilibrium points by the location of roots of the characteristic equation in the complex plane. Equilibrium points are classified according to the type of trajectory as focus, node, saddle, vortex and center.

Classical trajectories are not observable for pure DC circuits. However, so-called virtual trajectories can be studied using the thermodynamic approach. They illustrate how real fluctuations fade away or propagate from equilibrium. The form of virtual trajectories is given by the location of eigenvalues of the virtual matrix. Some virtual trajectories are given in [12].

On account of these analogies, we can state the following definition of DC stability:

- ✓ *DC stability is the ability of a system to dissolve virtual deflections from the operating point along the virtual trajectory.*

Then we can formulate a DC analogy of the First Lyapunov method:

- ✓ *If all eigenvalues of the virtual matrix of linearized system lie in the right-side complex plane, the real system is locally DC stable.*
- ✓ *If even only a sole eigenvalue of the virtual matrix lies in the left-side complex plane, the real system is DC unstable.*
- ✓ *If an eigenvalue of the virtual matrix lies in the imaginary axis, we cannot decide about the DC stability of real system by means of the linearized model.*

DC analogy of the Second Lyapunov method

The basic idea of the second Lyapunov method is as follows: For a stable system, the value of the Lyapunov function, which increases depending on the growing distance from the equilibrium point, has to decrease monotonically during a motion. Then we can easily follow the tendency of each trajectory by using the value of a single scalar function.

Now let us try to find a DC equivalent of the second Lyapunov function:

- ✓ *A system is asymptotically DC stable if it is possible to find such a definite Lyapunov function of the state that its derivative along the virtual trajectory is also definite but with inverse sign. The semidefinite derivative means mere DC stability.*
- ✓ *A system is DC unstable, if the derivative of the Lyapunov function along the virtual trajectory is definite, and the Lyapunov function in an arbitrary small region around the equilibrium state has the same sign as its derivative.*

Regarding a DC circuit as a thermodynamic system results in the following statement:

- ❢ *The Lyapunov function of a DC circuit is its dissipative potential.*

Equation (3) leads to a quadratic form, which gives a good idea of the course of dissipative potential in the neighbourhood of equilibrium point. The virtual matrix is the core of this quadratic form.

COMPUTER SIMULATIONS

Finding operating points

The Band-Gap cell by Brokaw [17] is in Fig. 3. Computer simulator found three operating points. Their list is in Tab. 1 (node 6 is connected to supply voltage +10V).

node	*voltage*	*voltage*	*voltage*
1	9.993V	9.950V	10.0V
2	601.3mV	770.4mV	0.0V
3	22.1mV	142.0mV	0.0V
4	9.999V	9.958V	10.0V
5	11.9mV	92.1mV	0.0V
6	10.0V	10.0V	10.0V

TABLE 1 - Three DC operating points.

Transient analysis

Transient analysis proves that operating points with output voltages of 770.4mV and 0V are stable, operating point with an output voltage of 601.3 mV is unstable. Trajectories are repelled from this point, even if the initial deflection from the equilibrium is very small and it proceeds slowly.

It is interesting that in the DC analysis using the circuit simulator, only the DC *unstable* operating point has been computed. The stable operating points has been located by the transient analysis.

The existence of DC unstable point is not apparent at first sight. It ensues from a detailed analysis of the relations on transistor bases according to the Fig. 4, where collector currents versus base voltage are plotted.

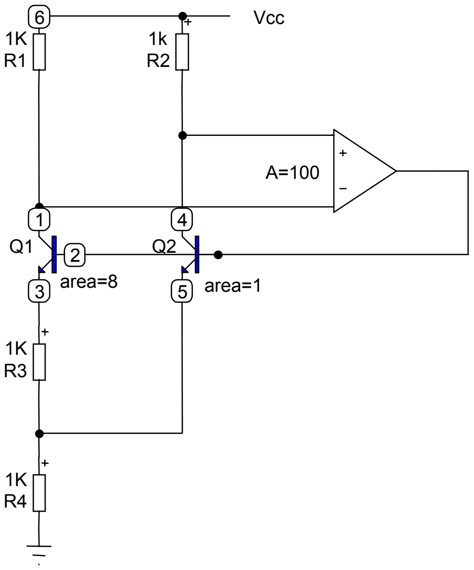

Fig. 2. *Band-Gap reference.*

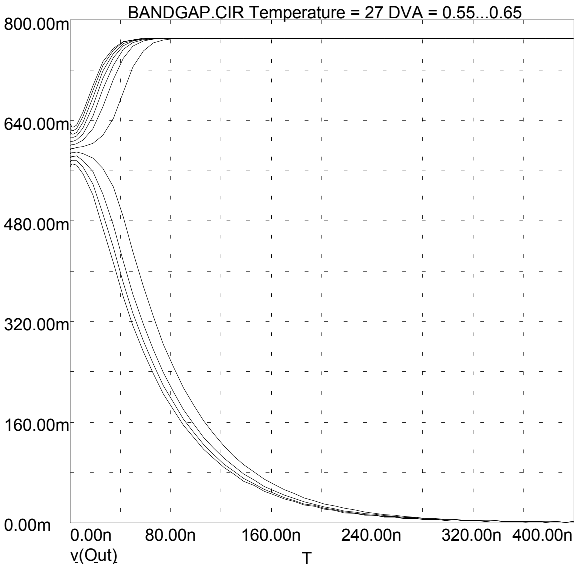

Fig. 3. *Output voltage of the Band-Gap cell.*

Transistor Q_1 has an 8 times larger area than transistor Q_2. That is why it opens “readily”. This advantage is balanced by the globally bigger emitter resistance, so that both characteristics intersect for a base voltage of about 790mV. The base voltage serves simultaneously as the output voltage of error amplifier for the magnification of deviation, which is proportional to the difference of collector currents.

The stable operating point with an output voltage of 770.4 mV is stabilized by the following mechanism: upward fluctuation of this voltage decreases the difference of collector currents. That is why voltage on the differential amplifier input decreases. Fluctuation is equalized by decreasing the base voltage.

From the shape of the characteristics, we can also analyze the stability of operating point with a base voltage of 0V, and the instability of operating point with a base voltage of 601.3 mV. The DC instability is caused by the shape of nonlinearity.

The DC unstable point affects the circuit dynamics if the circuit trajectory gets close to it, for instance after switching the sources, as shown in Fig. 5.

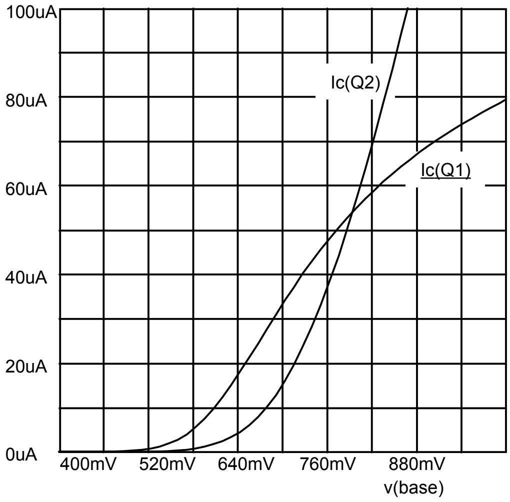

Fig. 4. *Nonlinearity as a cause of DC instability.*

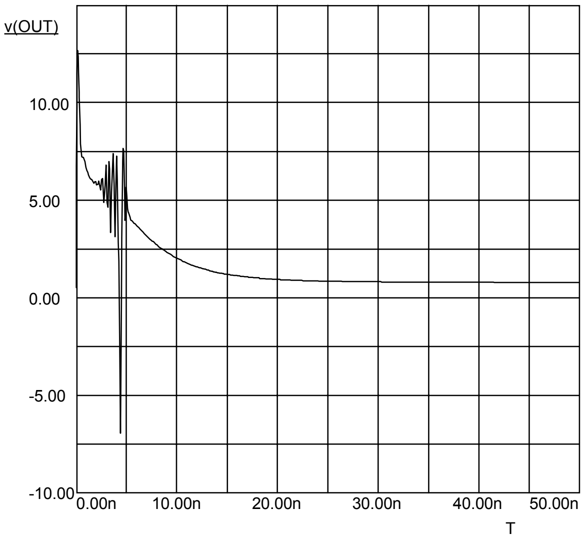

Fig. 5. *Actual settling of voltage reference to 770.4 mV.*

The hidden DC unstable point manifests itself during the circuit transient, when –in combination with dynamics proper– the trajectory has to overcome the local maximum of dissipative potential function. The repulsive force of this point is also manifested during a slow transient, when the supply voltage increases gradually and the system comes to the opposite stable point, i.e. with zero output voltage.

Frequency analysis

The so-called immitance criterion [18] can be used to test the stability of each equilibrium state separately.

Frequency responses of loading admittance of the fluctuation source are in Fig. 6. This source is placed into the output of error amplifier.

The curve for an output voltage of 601 mV starts on the negative real axis. It implies negative DC conductivity and thus DC instability. The other two curves imply both the DC and classical stability. Though the curve for output voltage starts at the origin of coordinates, stability can be inferred from further response for higher frequencies (this is not given in Fig. 6). We make use of the full version of immitance criterion [18]:

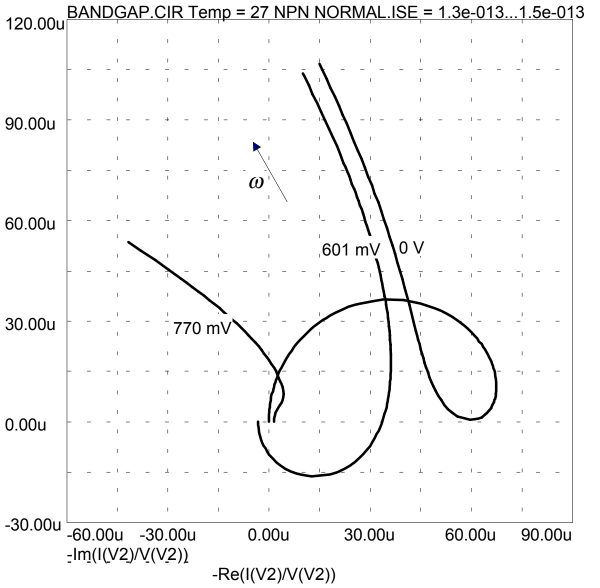

Fig. 6. *Frequency dependence of the amplifier output conductance.*

✓ *Let $X(j\omega)$ be an immitance into which a source of fluctuation works. Equilibrium state will be stable with respect to this fluctuation if the following conditions are fulfilled simultaneously:*

- $\lim\limits_{\omega \to 0} Re\left\{X(j\omega)\right\} > 0$,
- *hodograph of function* $X(j\omega)$ *for* $\omega \in \langle -\infty, \infty)$ *circles the origin of the complex plane clockwise just N - times, where N is the number of zeros of immitance* $X(s)$ *in the right-side complex plane.*

Note that stability is generally examined with respect to a concrete fluctuation. It again holds that improper selection of fluctuation sources can lead to an insufficient examination of the state space. Then the criterion need not warrant the necessary and also sufficient stability condition.

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THE AUTHORS

Dr. Zdenek Biolek and Prof. Dalibor Biolek are with the SPSE Roznov p.R. and with the Department of Telecommunications Brno University of Technology, Czech Republic, respectively.

E-mail: biolek@spseroznov.cz
dalibor.biolek@vabo.cz

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